

**Turbulent Flows**  
Stephen B. Pope  
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**Solution to Exercise 10.3**

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- a) Consider the log-law region of a wall-bounded turbulent flow. The turbulent viscosity hypothesis say

$$-\langle uv \rangle = \nu_T \frac{\partial \langle U \rangle}{\partial y}. \quad (1)$$

With  $\nu_T = ck^{1/2}\ell_m$ , we get

$$\langle uv \rangle = -ck^{1/2}\ell_m \frac{\partial \langle U \rangle}{\partial y}. \quad (2)$$

According to the log-law

$$u^+ = \frac{1}{\kappa} \ln y^+ + B \quad (3)$$

where  $y^+ = \frac{y}{\delta_\nu} = \frac{u_\tau y}{\nu}$ ,  $u_\tau = \sqrt{\frac{\tau_w}{\rho}}$ ,  $\delta_\nu = \frac{\nu}{u_\tau}$  and  $u^+ = \frac{\langle U \rangle}{u_\tau}$ , we get

$$\begin{aligned} \frac{\partial \langle U \rangle}{\partial y} &= \frac{u_\tau^2}{\nu} \frac{du^+}{dy^+} = \frac{u_\tau^2}{\nu \kappa y^+} \\ &= \frac{u_\tau}{\kappa y}. \end{aligned} \quad (4)$$

Substituting Eq. 4 into Eq. 2, we get

$$\langle uv \rangle = -ck^{1/2}\ell_m u_\tau / (\kappa y). \quad (5)$$

In the log-law region of a wall-bounded turbulent flow,  $\langle uv \rangle \approx -\tau_w / \rho$ , so

$$u_\tau = |\langle uv \rangle|^{1/2}. \quad (6)$$

Substituting Eq. 6 into Eq. 5 and using  $\ell_m = \kappa y$ , we get

$$c = |\langle uv \rangle / k|^{1/2}. \quad (7)$$

And in the log-law region of a wall-bounded flow,  $c \approx 0.55$ .

b) Given  $\mathcal{P} = \varepsilon$  and  $\mathcal{P} = -\langle uv \rangle \frac{\partial \langle U \rangle}{\partial y}$ , we get

$$-\langle uv \rangle \frac{\partial \langle U \rangle}{\partial y} = \varepsilon. \quad (8)$$

Substituting Eq. 4 into 8 and using Eq. 6, we get

$$\varepsilon = |\langle uv \rangle| \frac{u_\tau}{\kappa y} = \frac{|\langle uv \rangle|^{3/2}}{\kappa y}. \quad (9)$$

Using Eq.7 and  $\ell_m = \kappa y$ , we finally get

$$\varepsilon = \frac{c^3 k^{3/2}}{\ell_m}. \quad (10)$$

c) With  $\varepsilon = \frac{c^3 k^{3/2}}{\ell_m}$ , we get

$$\nu_T = ck^{1/2} \ell_m = ck^{1/2} c^3 k^{3/2} / \varepsilon = c^4 k^2 / \varepsilon. \quad (11)$$

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