Turbulent Flows
Stephen B. Pope
Cambridge University Press (2000)

Solution to Exercise 10.3

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a) Consider the log-law region of a wall-bounded turbulent flow. The turbulent viscosity hypothesis say

$$-\langle uv\rangle = \nu_T \frac{\partial \langle U\rangle}{\partial y}. \tag{1}$$

With $\nu_T = ck^{1/2}\ell_m$, we get

$$\langle uv\rangle = -ck^{1/2}\ell_m \frac{\partial \langle U\rangle}{\partial y}. \tag{2}$$

According to the log-law

$$u^+ = \frac{1}{\kappa}\ln y^+ + B \tag{3}$$

where $y^+ = \frac{y}{\delta_\nu} = \frac{u_\tau y}{\nu}$, $u_\tau = \sqrt{\frac{\tau_w}{\rho}}$, $\delta_\nu = \frac{\nu}{u_\tau}$ and $u^+ = \frac{\langle U\rangle}{u_\tau}$, we get

$$\begin{aligned} \frac{\partial \langle U\rangle}{\partial y} &= \frac{{u_\tau}^2}{\nu}\frac{du^+}{dy^+} = \frac{{u_\tau}^2}{\nu\kappa y^+} \\ &= \frac{u_\tau}{\kappa y}. \end{aligned} \tag{4}$$

Substituting Eq. 4 into Eq. 2, we get

$$\langle uv\rangle = -ck^{1/2}\ell_m u_\tau/(\kappa y). \tag{5}$$

In the log-law region of a wall-bounded turbulent flow, $\langle uv\rangle \approx -\tau_w/\rho$, so

$$u_\tau = |\langle uv\rangle|^{1/2}. \tag{6}$$

Substituting Eq. 6 into Eq. 5 and using $\ell_m = \kappa y$, we get

$$c = |\langle uv \rangle / k|^{1/2}. \tag{7}$$

And in the log-law refion of a wall-bounded flow, $c \approx 0.55$.

b) Given $\mathcal{P} = \varepsilon$ and $\mathcal{P} = -\langle uv \rangle \frac{\partial \langle U \rangle}{\partial y}$, we get

$$-\langle uv \rangle \frac{\partial \langle U \rangle}{\partial y} = \varepsilon. \tag{8}$$

Substituting Eq. 4 into 8 and using Eq. 6, we get

$$\varepsilon = |\langle uv \rangle| \frac{u_\tau}{\kappa y} = \frac{|\langle uv \rangle|^{3/2}}{\kappa y}. \tag{9}$$

Using Eq.7 and $\ell_m = \kappa y$, we finally get

$$\varepsilon = \frac{c^3 k^{3/2}}{\ell_m}. \tag{10}$$

c) With $\varepsilon = \frac{c^3 k^{3/2}}{\ell_m}$, we get

$$\nu_T = c k^{1/2} \ell_m = c k^{1/2} c^3 k^{3/2} / \varepsilon = c^4 k^2 / \varepsilon. \tag{11}$$